ON THE CONVERGENCE OF THE STURM-LIOUVILLE SERIES.*

By J. L. Walsh.

1. We shall consider in this paper the differential equation

$$u''(x) + [\varrho^2 - g(x)]\, u(x) = 0, \qquad 0 \leqq x \leqq 1, \tag{1}$$

where ϱ is a parameter, in connection with the boundary conditions

$$u(0) = 0, \qquad u(1) = 0. \tag{2}$$

For certain values of ϱ, the so-called characteristic values, equation (1) has solutions (normal solutions or characteristic functions) which satisfy (2), and in mathematical physics there arises the problem of the development of arbitrary functions in series in terms of these normal solutions. These series are a special, but important and typical, case of the more general Sturm-Liouville series which are the developments of arbitrary functions in terms of normal solutions of equations similar to (1) and which satisfy homogeneous boundary conditions similar to (2).

Under certain restrictions on $g(x)$, we shall prove that, for any function on the interval $0 \leqq x \leqq 1$ integrable in the sense of Lebesgue and with an integrable square, the series which is the formal expansion in terms of the normal functions which correspond to (1), (2) has essentially the same convergence properties as the series which is the formal expansion in terms of the normal functions which correspond to (1), (2) when $g(x) \equiv 0$. This last set of functions is, except for a constant factor, the set $\{\sin k\pi x\}$, and the convergence properties of the expansions of arbitrary functions in terms of this set are well known.

2. The results, but not the methods, of the present paper are closely connected with the work of Haar† although he considers the boundary conditions

$$u'(0) - hu(0) = u'(1) + Hu(1) = 0 \tag{3}$$

* Presented to the American Mathematical Society, December, 1920. It is largely due to Dr. T. H. Gronwall that the result here published has its present comparatively simple form. Thus Dr. Gronwall eliminated from the hypothesis of the principal theorem an unnecessary assumption and indicated to the writer more than a corresponding simplification of the use of the asymptotic expressions.

† Math. Annalen, vol. 69 (1910), pp. 331–371; vol. 71 (1912), pp. 38–53.

instead of conditions (2) in connection with equation (1). Thus Haar proves that the Sturm-Liouville series of an integrable function is convergent, divergent, or summable at a point according as its Fourier cosine series is convergent, divergent, or summable at that point; the present paper deals merely with functions integrable and with an integrable square. Haar does not, however, bring out clearly the identity of the properties of uniform convergence for the two developments,* nor is it obvious how his methods can be extended to include absolute convergence; the present paper deals with both uniform and absolute convergence. Haar makes use of the fact that an analytic function can be approximated as closely as desired by a linear combination of normal functions; the present paper proves at a single step the possibility of expansion of all functions (integrable and with an integrable square) which can be developed into a Fourier sine series, and the identity of the convergence properties of the two developments.

The similarity of uniform convergence for the two expansions, in the precise manner which appears later, enables us to state that Gibbs' phenomenon occurs for our Sturm-Liouville series in precisely the same manner as for the Fourier sine series. Gibbs' phenomenon for the Sturm-Liouville series seems first to have been pointed out by Weyl.†

3. We shall first prove a general theorem concerning expansions of arbitrary functions and then apply that theorem to the system (1), (2).

A set of functions $\{u_n(x)\}$ $(n = 1, 2, 3, \ldots)$ continuous on the interval $0 \leqq x \leqq 1$ is said to be *normal* on that interval if and only if

$$\int_0^1 u_n^2(x)\,dx = 1, \qquad (n = 1, 2, 3, \ldots,),$$

and is said to be *orthogonal* if and only if

$$\int_0^1 u_i(x)\,u_j(x)\,dx = 0, \qquad (i \neq j).$$

We shall consider, under certain restrictions, two normal orthogonal sets of functions $\{u_n(x)\}$ and $\{\overline{u}_n(x)\}$ and shall suppose that the $\{\overline{u}_n(x)\}$ can be

* For uniform convergence and the developments of *continuous* functions, see the comment on Haar's work by Bôcher, Proceedings of the Fifth International Congress of Mathematicians (Cambridge), vol. I, pp. 191, 192.

Compare also the reference given in the next following footnote.

† For equation (1) in connection with the boundary conditions $u'(0) = u'(1) = 0$, see Rend. Circ. Mat. Palermo, vol. 29 (1910), pp. 321-323.

expanded in terms of the $\{u_n(x)\}$. We invert this system of equations and thereby express the $\{u_n(x)\}$ in terms of the $\{\bar{u}_n(x)\}$. To prove that the convergence properties of the formal expansion of an arbitrary function $f(x)$ in terms of the $\{\bar{u}_n(x)\}$ are essentially the same as the convergence properties of the expansion in terms of the $\{u_n(x)\}$ (these properties are assumed to be known), in the former expansion we merely substitute the expansion of the $\{\bar{u}_n(x)\}$ in terms of the $\{u_n(x)\}$. Rearrangement of the terms then gives us precisely the expansion of $f(x)$ in terms of the $\{u_n(x)\}$. The following exposition will not seem unnatural if this general method is kept in mind.

4. To prepare for the inversion of the system of equations indicated, we now prove

LEMMA I. *If the set of real numbers* $\{c_{nk}\}$, $n, k = 1, 2, 3, \ldots$, *is such that*

$$c_{nk} + c_{kn} + \sum_{\nu=1}^{\infty} c_{n\nu}\, c_{k\nu} = 0, \tag{4}$$

and if the series

$$\sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{\infty} |c_{nk}| \right\}^2 \tag{5}$$

converges, then we have also

$$c_{kn} + c_{nk} + \sum_{\nu=1}^{\infty} c_{\nu n}\, c_{\nu k} = 0. \tag{6}$$

We shall have frequent occasion to use the Lagrange inequality which holds for any two sets of real numbers $\{a_\nu\}$ and $\{b_\nu\}$:

$$\left(\sum_{\nu=1}^{m} |a_\nu\, b_\nu| \right)^2 \leqq \sum_{\nu=1}^{m} a_\nu^2 \sum_{\nu=1}^{m} b_\nu^2,$$

whence the well known and frequently used fact that, when $\sum\limits_{\nu} a_\nu^2$ and $\sum\limits_{\nu} b_\nu^2$ both converge, then also $\sum\limits_{\nu} |a_\nu\, b_\nu|$ converges and

$$\left(\sum_{\nu} |a_\nu\, b_\nu| \right)^2 \leqq \sum_{\nu} a_\nu^2 \sum_{\nu} b_\nu^2. \tag{7}$$

Here and below, unless otherwise stated, summation subscripts run from 1 to ∞.

From the convergence of (5), then, follows immediately the convergence of the triple series

(8) $$\sum_{\mu\nu n} |c_{n\mu}\, c_{n\nu}|.$$

All terms of the series

(9) $$\sum_{nk} c_{nk}^2,$$

(10) $$\sum_{\nu} c_{n\nu}^2,$$

are contained in (8), so these two series converge. The absolute convergence of the series contained in (4) follows by application of (7). The absolute convergence of the series contained in (6) follows from the convergence of (8).

Denote by ε_{nk} the left-hand member of (6), so that we have

(11) $$\varepsilon_{nk}^2 = (c_{nk} + c_{kn})^2 + 2\sum_{\nu} c_{nk}\, c_{\nu n}\, c_{\nu k} + 2\sum_{\nu} c_{kn}\, c_{\nu n}\, c_{\nu k} + \sum_{\mu\nu} c_{\mu n}\, c_{\mu k}\, c_{\nu n}\, c_{\nu k}.$$

Write

(12) $$s = \sum_{nk} (c_{nk} + c_{kn})^2;$$

the convergence of (12) follows from the convergence of (9) and the general inequality

(13) $$(a + b)^2 \leqq 2a^2 + 2b^2.$$

The series $\sum_{\nu nk} c_{nk}\, c_{\nu n}\, c_{\nu k}$ converges absolutely; for, by the convergence of (9), $|c_{nk}|$ remains less than some positive M, we have

$$|c_{nk}\, c_{\nu n}\, c_{\nu k}| < M\, |c_{\nu n}\, c_{\nu k}|,$$

and we know that (8) converges. Then from (4) we have

$$\sum_{\nu nk} c_{nk}\, c_{\nu n}\, c_{\nu k} = \sum_{\nu n} c_{\nu n} \sum_{k} c_{nk}\, c_{\nu k} = -\sum_{\nu n} c_{\nu n}\,(c_{\nu n} + c_{n\nu}).$$

Similarly we have

$$\sum_{\nu nk} c_{kn}\, c_{\nu n}\, c_{\nu k} = \sum_{\nu k} c_{\nu k} \sum_{n} c_{kn}\, c_{\nu n} = -\sum_{\nu k} c_{\nu k}\,(c_{\nu k} + c_{k\nu}) = -\sum_{\nu n} c_{n\nu}\,(c_{n\nu} + c_{\nu n});$$

the last step is simply a change of notation. By (12) we now find

(14) $$\sum_{\nu nk} c_{nk}\, c_{\nu n}\, c_{\nu k} + \sum_{\nu nk} c_{kn}\, c_{\nu n}\, c_{\nu k} = -s.$$

The quadruple series $\sum_{\mu\nu nk} |c_{\mu n}\, c_{\mu k}\, c_{\nu n}\, c_{\nu k}|$ converges; for, by the convergence of (8), $\sum_{\mu} |c_{\mu n}\, c_{\mu k}|$ for all n and k remains less than some positive M_1, we have

$$\sum_{\mu} |c_{\mu n}\, c_{\mu k}\, c_{\nu n}\, c_{\nu k}| < M_1 |c_{\nu n}\, c_{\nu k}|,$$

and $\sum_{\nu nk} |c_{\nu n}\, c_{\nu k}|$ is (8) and known to be convergent. A further use of (4) gives the result

$$(15)\quad \begin{aligned} \sum_{\mu\nu nk} c_{\mu n}\, c_{\mu k}\, c_{\nu n}\, c_{\nu k} &= \sum_{\mu\nu n} c_{\mu n}\, c_{\nu n} \sum_{k} c_{\mu k}\, c_{\nu k} = -\sum_{\mu\nu n} c_{\mu n}\, c_{\nu n}\, (c_{\mu\nu} + c_{\nu\mu}) \\ &= -\sum_{\mu\nu} (c_{\mu\nu} + c_{\nu\mu}) \sum_{n} c_{\mu n}\, c_{\nu n} = \sum_{\mu\nu} (c_{\mu\nu} + c_{\nu\mu})^2 = s. \end{aligned}$$

From (11), (12), (14) and (15) we conclude

$$\sum_{nk} \varepsilon_{nk}^2 = s - 2s + s = 0,$$

so that every ε_{nk} vanishes and Lemma I is established.

5. We shall apply Lemma I to prove

LEMMA II. *If $\{u_n(x)\}$ and $\{\bar{u}_n(x)\}$ are two sets of functions normal and orthogonal on the interval $0 \leqq x \leqq 1$, if the former set is uniformly bounded on this interval, and if*

$$(16)\qquad \bar{u}_n(x) - u_n(x) = \sum_{k} c_{nk}\, u_k(x), \qquad (n = 1, 2, 3, \ldots),$$

where

$$c_{nk} = \int_0^1 (\bar{u}_n(x) - u_n(x))\, u_k(x)\, dx,$$

and where the series

$$(5)\qquad \sum_{n} (\sum_{k} |c_{nk}|)^2$$

converges, then the latter set of functions is also uniformly bounded on the interval and we have the developments

$$(17)\qquad u_n(x) - \bar{u}_n(x) = \sum_{k} c_{kn}\, \bar{u}_k(x), \qquad (n = 1, 2, 3, \ldots).$$

The two series (16) *and* (17) *converge absolutely and uniformly on the interval.*

The series $\sum_k |c_{nk}|$ converges and is uniformly bounded for all n by the convergence of (5), so the series (16) converges absolutely and uniformly and the set $\{\bar{u}_n\}$ is uniformly bounded on the interval.

In the normality-orthogonality condition

$$\int_0^1 \bar{u}_n(x)\,\bar{u}_k(x)\,dx = \begin{cases} 1, & n = k, \\ 0, & n \neq k, \end{cases}$$

we insert the absolutely and uniformly convergent series (16) for the $\{\bar{u}_n\}$, multiply out and integrate term by term, observing that the $\{u_n\}$ are normal and orthogonal. The result is (4), and (6) follows by Lemma I. From the convergence of (8) and the properties of $\sum_k |c_{nk}|$ follow the convergence and the uniform boundedness for all n of $\sum_k |c_{nk}|$. Thus the right-hand member of (17) converges absolutely and uniformly.

From the relation (6) multiplied by u_k and summed for all k we have, making use of (16) and the convergence of $\sum_{\nu k} |c_{\nu n}\, c_{\nu k}|$ proved from the convergence of (8),

$$\sum_k c_{kn}\, u_k + \sum_k c_{nk}\, u_k + \sum_\nu c_{\nu n} \sum_k c_{\nu k}\, u_k = 0,$$

$$\sum_k c_{kn}\, u_k + (\bar{u}_n - u_n) + \sum_\nu c_{\nu n}\,(\bar{u}_\nu - u_\nu) = 0;$$

this last equation becomes essentially (17) if we identify ν with k.

6. We are now in a position to prove our principal theorem.

THEOREM. *Let $\{u_n(x)\}$ and $\{\bar{u}_n(x)\}$ be two sets of functions normal and orthogonal on the interval $0 \leq x \leq 1$, the former set uniformly bounded on this interval, and such that*

$$\bar{u}_n(x) - u_n(x) = \sum_{k=1}^{\infty} c_{nk}\, u_k(x), \quad (n = 1, 2, 3, \ldots), \tag{16}$$

where

$$c_{nk} = \int_0^1 [\bar{u}_n(x) - u_n(x)]\, u_k(x)\, dx$$

and where the series

$$\sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{\infty} |c_{nk}| \right\}^2 \tag{5}$$

converges. Then, if $f(x)$ is any function integrable and with an integrable square, $0 \leq x \leq 1$, the two series

$$(18) \qquad f(x) \backsim \sum_{k=1}^{\infty} a_k u_k(x),$$

$$(19) \qquad f(x) \backsim \sum_{k=1}^{\infty} b_k \bar{u}_k(x)$$

where

$$a_k = \int_0^1 f(x) u_k(x)\, dx, \qquad b_k = \int_0^1 f(x) \bar{u}_k(x)\, dx,$$

have essentially the same convergence properties; this in the sense that the series corresponding to their term-by-term difference

$$(20) \qquad \sum_{k=1}^{\infty} (a_k u_k(x) - b_k \bar{u}_k(x))$$

converges absolutely and uniformly on the entire interval to the sum zero.

The sign $\backsim$ is used simply to denote that the coefficients a_k and b_k are given by the formulas indicated. Of course, if $f(x)$ is equal to the series (18), for example, and if we are at liberty to multiply both members of the equation by $u_k(x)$ and to integrate term by term the resulting series, a_k must be given by the formula indicated.

We shall use the two relations obtained by term-by-term integration of (16) and (17) after multiplication by $f(x)\, dx$; this formal work is justified by the convergence of $\sum_k |c_{nk}|$ and $\sum_k |c_{kn}|$,

$$(21) \qquad b_n - a_n = \sum_k c_{nk} a_k,$$

$$(22) \qquad a_n - b_n = \sum_k c_{kn} b_k.$$

By (22) and (16), (20) is equivalent to

$$(23) \quad \sum_k [(a_k - b_k) u_k - b_k (\bar{u}_k - u_k)] = \sum_k \left[\sum c_{\nu k} b_\nu u_k - \sum_\nu c_{k\nu} b_k u_\nu\right].$$

The convergence of $\sum_n b_n^2$ (and similarly of $\sum_n a_n^2$) follows from the Bessel inequality

$$(24) \qquad \int_0^1 f^2(x)\, dx \geqq \sum_n b_n^2,$$

which may be proved by the relation

$$\int_0^1 \left[f(x) - \sum_{n=1}^{N} b_n \bar{u}_n(x) \right]^2 dx \geqq 0.$$

The convergence of

$$\sum_n |b_n| \sum_k |c_{nk}| \tag{25}$$

now follows from (7) and the convergence of (5), and proves the absolute and uniform convergence of the right-hand member of (23) and the theorem.

7. It follows immediately that *properties of absolute convergence, convergence, summability, and divergence at any given point obtain for one of the series* (18), (19) *as for the other, and likewise the properties of uniform convergence in the entire interval considered or in any sub-interval, uniform summability, and also term-by-term integrability. Whenever the two series are convergent, summable, or properly divergent, their sums are the same.* The nature of the approximating functions and of their approach to the limit (in the case of convergence) at a point of continuity or of discontinuity of $f(x)$ is essentially the same for (18) as for (19). In particular, if Gibbs' phenomenon occurs for (18), it also occurs for (19). The reader will notice various other properties* common to the sets $\{u_n\}$ and $\{\bar{u}_n\}$, such as the existence or non-existence of a continuous function for which the formal series does not converge at every point.

If there exists no continuous function $f(x)$, not identically zero, such that all the $\{a_k\}$ are zero, the set $\{u_k\}$ is said to be *closed* with respect to continuous functions.† We have assumed nothing of the closure of $\{u_k\}$ or $\{\bar{u}_k\}$, but it results from (21) and (22) that, if either set is closed, so is also the other.

8. The closure of the set of characteristic functions of the system (1), (2) has recently been proved by Professor Birkhoff‡ from the closure of the set for the system (1), (2) when $g(x) \equiv 0$. It was Professor Birkhoff's note which suggested to me the possibility of the present treatment.

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* Thus the simultaneous convergence or divergence of $\sum_n |a_n|$ and $\sum_n |b_n|$ follows from (22) and the convergence of (25).

† There is a corresponding definition and similarity of the property for $\{u_k\}$ and $\{\bar{u}_k\}$ for functions $f(x)$ integrable and with an integrable square.

‡ Proc. Nat. Acad. Sci., vol. 3 (1917), pp. 656-659. The necessary facts concerning the asymptotic nature of the characteristic functions were later proved in detail by Murray, these Annals, ser. (2), vol. 22 (1920-1921), pp. 145-156. The latter paper contains a far more detailed investigation than is necessary for the application of our general theorem.

The methods and general theorem of the present paper are similar to the methods and general theorem of a recent paper* in which there is considered the generalization of a normal and orthogonal system $\{u_k\}$ to a system $\{\bar{u}_k\}$ which is not necessarily normal and orthogonal.

Application to Sturm-Liouville Series.

9. We now apply our main theorem (§ 6) to the case where

$$u_k(x) = \sqrt{2}\sin k\pi x \tag{26}$$

and $\bar{u}_k(x)$ is the k^{th} characteristic function of the system (1), (2). We assume for convenience that $g(x)$ is continuous† on the interval $0 \leqq x \leqq 1$.

We notice directly from the differential equation (2) that $\bar{u}_k(x)$ is a function with a continuous second derivative, so the most elementary theory of Fourier's series informs us that the development (16) is valid. It remains merely to prove the convergence of (5); for this proof we shall need only those most simple asymptotic formulas for $\bar{u}_k(x)$ and ϱ_k (the k^{th} characteristic number of the system (1), (2)) which are readily proved by the original method of Liouville‡:

$$\varrho_k - k\pi \to 0 \text{ as } k \to \infty, \tag{27}$$

$$\bar{u}_k(x) = \sqrt{2}\left(\sin k\pi x + \frac{1}{k}\varphi_k(x)\right), \qquad |\varphi_k(x)| < c, \tag{28}$$

for all k and all x on the interval. It is interesting to note that Haar needs the expansion of ϱ_k and $\bar{u}_k(x)$ up to a remainder term of the order $1/n^2$.

The notation for c_{nk} gives us the formula

$$c_{nk} + \delta_{nk} = \int_0^1 \bar{u}_n(x) \cdot \sqrt{2}\sin k\pi x\,dx, \qquad \delta_{nk} = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases} \tag{29}$$

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* Walsh, A generalization of the Fourier cosine series, Trans. Amer. Math. Soc., vol. 22 (1921), pp. 230–239.

† Even this restriction may be lightened with little difficulty. Haar (l. c.) uses the asymptotic expansion given by Hobson, and Hobson supposes $g(x)$ to be of bounded variation.

‡ See, for example, Kneser, Integralgleichungen, pp. 95-98.

If we integrate twice by parts and make use of the boundary conditions (2) for $\overline{u}_n(x)$, we have

$$(30)\qquad c_{nk}+\delta_{nk} = -\frac{1}{k^2\pi^2}\int_0^1 \overline{u}_n''(x)\cdot\sqrt{2}\sin k\pi x\,dx.$$

By (27) there exists a number N such that

$$(31)\qquad |\varrho_n - n\pi| < \frac{1}{2} \text{ for } n > N.$$

We have by (30) [c_1, c_2, etc. denote positive constants],

$$|c_{nk}| < \frac{c_1}{k^2} \text{ for } k \neq n, \quad n \leqq N,$$

$$|c_{nn}| < 1 + \frac{c_1}{n^2} \text{ for } n \leqq N;$$

the boundedness of $\overline{u}_n''(x)$ follows from the boundedness of the $\{\overline{u}_n(x)\}$ and equation (1). Hence $\sum_k |c_{nk}|$ converges for every n, and

$$\sum_{n\leqq N}\Big\{\sum_k |c_{nk}|\Big\}^2$$

has a finite value.

10. To prove the convergence of

$$\sum_{n>N}\Big\{\sum_k |c_{nk}|\Big\}^2,$$

we make $k = n$ in (29) and make use of (28):

$$c_{nn}+1 = 1+\int_0^1 \frac{2}{n}\varphi_n(x)\sin n\pi x\,dx,$$

so that

$$(32)\qquad |c_{nn}| < \frac{c_2}{n}.$$

For $k \neq n$, we substitute the value of $\bar{u}_n''(x)$, found from the differential equation (1), in (30), whence

$$c_{nk} = \frac{\varrho_n^2}{k^2\pi^2}\int_0^1 \bar{u}_n(x)\sqrt{2}\sin k\pi x\,dx - \frac{1}{k^2\pi^2}\int_0^1 g(x)\bar{u}_n(x)\sqrt{2}\sin k\pi x\,dx.$$

The first integral on the right is precisely c_{nk}, by (29), so we have

$$(33)\qquad (\varrho_n^2 - k^2\pi^2)\,c_{nk} = \int_0^1 g(x)\,\bar{u}_n(x)\sqrt{2}\sin k\pi x\,dx.$$

We are supposing $n > N$ and $k \neq n$, so that $|n-k| \geqq 1$, and we have by (31)

$$|\varrho_n - k\pi| = |\varrho_n - n\pi + (n-k)\pi| \gtreqless |n-k|\pi - |\varrho_n - n\pi| > |n-k|\pi - \frac{1}{2} > |n-k|,$$

$$|\varrho_n + k\pi| = |\varrho_n - n\pi + (n+k)\pi| \gtreqless (n+k)\pi - |\varrho_n - n\pi| > n+k.$$

Then (33) gives the inequality

$$(34)\qquad |c_{nk}| < \frac{1}{|k^2-n^2|}\left|\int_0^1 g(x)\,\bar{u}_n(x)\sqrt{2}\sin k\pi x\,dx\right|.$$

11. From the identity

$$\frac{1}{k^2-n^2} = \frac{1}{2n}\,\frac{1}{k-n} - \frac{1}{2n}\,\frac{1}{k+n}$$

we find by (13) that

$$\frac{1}{(k^2-n^2)^2} \leqq \frac{1}{2n^2}\,\frac{1}{(k-n)^2} + \frac{1}{2n^2}\,\frac{1}{(k+n)^2}.$$

Moreover we have

$$\sum_k{}' \frac{1}{(k-n)^2} = \sum_{k=1}^{n-1}\frac{1}{(n-k)^2} + \sum_{k=n+1}^{\infty}\frac{1}{(k-n)^2} < 2\sum_k \frac{1}{k^2},$$

$$\sum_k{}' \frac{1}{(k+n)^2} < \sum_k \frac{1}{k^2},$$

where the accent on the summation sign indicates that the term for $k = n$ is to be omitted. We have finally the inequality

$$\sum_k{}' \frac{1}{(k^2-n^2)^2} < \frac{3}{2n^2} \sum_k \frac{1}{k^2} = \frac{c_3}{n^2}. \tag{35}$$

12. Let us now apply Bessel's inequality (24) to the function $g(x)\,\overline{u}_n(x)$; we obtain

$$\sum_k{}' \left(\int_0^1 g(x)\,\overline{u}_n(x) \sqrt{2} \sin k\pi x \, dx \right)^2 \leqq \int_0^1 g^2(x)\,\overline{u}_n^2(x)\, dx < c_4. \tag{36}$$

Apply inequality (7) to the series in (35) and (36); we have by (34) that

$$\left(\sum_k{}' |c_{nk}| \right)^2 < \frac{c_3 c_4}{n^2}. \tag{37}$$

Apply inequality (13) to (32) and the series of (37); we have

$$\left(\sum_k |c_{nk}| \right)^2 < \frac{2c_2^2 + 2c_3 c_4}{n^2} = \frac{c_5}{n^2}, \qquad n > N,$$

whence the convergence of

$$\sum_{n>N} \left(\sum_k |c_{nk}| \right)^2$$

follows immediately.